An asset allocation model with inequalities constraints and coherent risk measure: an application to Brazilian equities

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We propose a method for optimal portfolio selection built on the Black and Litterman model and with two major contributions. We introduce in the investor objective function a risk measure named expected tail loss, which is useful in portfolio selection context as it supports the benefits of diversification and we allow investor views to be expressed in terms of linear inequalities among expected returns, which seems more natural in the practice of portfolio selection. Further we implement the models using market database applied to Brazilian equities. The results show that our approach leads to lower risk optimal portfolios and that our proposed methodology to implement the investor subjective views led to optimal portfolios with superior outcomes.

Keywords: Black-Litterman, expected tail loss, portfolio optimization.

Um modelo de alocação de ativos com restrições de desigualdades e medidas de risco coerente: uma aplicação para ações brasileiras

Propomos um método para otimização de carteiras baseado no modelo proposto por Black e Litterman com duas contribuições relevantes. Introduzimos na função objetivo do investidor uma medida de risco denominada risco de cauda, que é bastante adequada no contexto de seleção de carteiras já que considera os princípios da diversificação. Permitimos neste estudo que os investidores expressem suas visões acerca dos ativos em termos de desigualdades lineares entre os retornos esperados, o que nos parece mais natural na aplicação prática de seleção de ativos. Os modelos foram implementados usando um banco de dados de ações brasileiras. Os resultados mostram que a nossa abordagem leva à alocação em carteiras ótimas com menor risco quando comparado aos modelos anteriores e que a metodologia proposta para expressar as opiniões dos investidores levaram a carteiras ótimas com resultados superiores.

Palavras-chave: Black-Litterman, risco de cauda, otimização de carteiras.

INTRODUCTION

Objective

The objective of this study is to propose an alternative asset allocation model that is robust to parameters uncertainties and estimation errors.

Relevance

The relevance of our methodology is illustrated by a portfolio selection experiment on the Brazilian equity market.

Methodology

Our model builds on the Black-Litterman model for portfolio selection, but allows views to be expressed in terms of linear inequalities among expected returns. Also, starting from the observation that positive and negative deviations of the returns from their mean usually play a great asymmetric role in the investor perception, we decided to use a quantile based measure in the objective function, namely the expected tail loss. This is a coherent risk measure which is important in portfolio selection as it supports the benefits of diversification.

ASSET ALLOCATION MODELS AND RELATED LITERATURE

In 1952 Harry Markowitz published the article “Portfolio Selection” which can be considered as the beginning of modern portfolio theory. Portfolio selection is the problem of allocating capital over a number of available assets in order to maximize the return on the investment while minimizing its risk. In a portfolio context, risk is usually measured by means of a dispersion measure, such as the variance or standard deviation (volatility) of returns around their expected value. The result of traditional portfolio optimization is a parabolic efficient frontier, indicating the combinations of assets with the highest expected return given a certain level of risk. Markowitz framework seems to be very reasonable in theory, but it continues to encounter skepticism among investment practitioners. One possible reason is the counter-intuitive nature of the optimal portfolios generated ([Michaud (1989), Black Litterman (1992)]). Empirical studies have shown that mean-variance optimal portfolio allocations (Markowitz’s framework) tend to concentrate on a small subset of the available securities and appear not to be well diversified ([Bera Park (2008)]). Furthermore, optimal portfolios are often sensitive to changes in the input parameters of the problem (expected returns and covariance matrix). Mean-variance optimization does not take into account the parameters uncertainty, which leads to very unstable results. Such observations indicate that the inputs to the mean-variance optimization model need to be very accurately estimated.[1][Jobson Korkie (1981), Michaud (1989), Best Grauer (1991), Chopra Ziemba (1993), Britten-Jones (1999)], among others, argued that the hypersensitivity of the optimal weights in the portfolio is the result of the nature of the errors of the mean-variance optimization.

The unstable results obtained by use of Markowitz model motivated Fisher Black and Robert Litterman (1992), then working at Goldman Sachs, to develop a new mean-variance model based on an Bayesian analytic framework ([Black Litterman (1992)]). The Black-Litterman (BL) model allows the investor to start with a prior belief about expected returns (subjective views) and to update this prior distribution with market empirical data (model-based estimates, such as CAPM-implied equilibrium returns as an approximation[2]). Nevertheless it is important to mention that since [Black Litterman (1992)] first presented their model, the CAPM has been rejected empirically ([Fama French (1992), Fama French (1993)]) and several asset pricing models, using a multi factor approach have been proposed. In this study our focus will be to propose an alternative investor view model and to implement different risk measures in the optimization problem. And beeing well aware of the limitations of using a misspecified asset pricing model to learn from market prices, we thus decided not to use any model of market equilibrium and to capture market expected returns from the weights of a well traded benchmark equity index in order to infer the expected returns posterior distribution.[3]

The investor views (prior distribution) are represented as linear combinations of estimates of asset returns. Each estimate must be provided with a measure of uncertainty associated with that particular view, which, given the assumption of normality, is chosen as the variance associated with the estimate. A remarkable feature of this approach is soundness. As the posterior expected returns are a combination of the prior investor views with the market equilibrium returns, in the absence of subjective views, the best strategy is to stick to the market via equilibrium views. On the other hand, if the investor has some views, the portfolio should be tilted to reflect these views combined. Since the market view is always considered, it is less likely to run into unstable or corner solutions. In case the investor holds some strong views that dominate the market view, the model also allows the results to be significantly adjusted towards these views. Based on these considerations, the BL model is appealing in theory and natural in practice.

Many studies further advanced the understanding and implementation of BL framework. [Lee (2000)] and [Satchell Scowcroft (2000)] elaborated and expanded the theoretical framework, while others, such

[1] Markowitz model needs expected returns estimates one step ahead which are usually assumed to be the average of historical returns, that is $E_t(r_{t+1}) \equiv \bar{r}$

[2] On the other hand, [Sharpe (1974)] proposed to extract expected returns from the weights of a portfolio held by an institution rather than from equilibrium weights of the market portfolio.

[3] Despite its popularity, we could not find other study that empirically evaluates the BL model for Brazilian market data. On the other hand, we can notice the model is becoming very popular for money management firms, beeing implemented by market data softwares (as Bloomberg, for instance), taught at financial schools and covered by investments textbooks.

as [Bevan Winkelmann (1998)], [He Litterman (1999)], [Herold (2003)], [Idzorek (2004)], and [Jones, Lim Zangari (2007)] focused on implementation.

The rest of the article is organized as follows. In section 2 we review the original BL methodology. In section 3 we describe our proposed method for portfolio choice problem addressing the expected tail loss framework. In section 4 we proceed with a case study applied to Brazilian equities and in section 5 we conclude.

Revisiting Black-Litterman asset allocation model

The Black-Litterman asset allocation model uses a Bayesian approach to infer the asset expected returns as random variables themselves.[4] In this study we will follow the approach proposed by [Satchell Scowcroft (2000)] and [Christodoulakis (2002)], which is consistent with the definition of Bayes Theorem.

Let us assume that there are N assets in the market, which may include equities, bonds, currencies etc. Unlike in classical statistics in which the means are considered deterministic (though unobservable), in the BL framework the actual mean in unknown and stochastic, although the covariance matrix of returns is considered well defined. The model applies the “known covariance unknown mean" Bayesian solution to statistical inference to generate expected asset returns suitable for use in the mean-variance Markowitz-type portfolio allocation. In essence, the approach hereafter consists of generating one-step-ahead posterior returns on the assets by means of a precision-matrix-weighted combination of the investor prior views[5] $\mathbf{q}$ of their future returns with the distribution of their implied returns $\boldsymbol{\pi}$ obtained by an equilibrium model (or reverse optimization from the historic covariance and the benchmark index portfolio of securities).

We consider that $\tilde{\mathbf{r}}$ is the vector of asset returns[6] with an unknown and stochastic mean $\tilde{\boldsymbol{\mu}}$[7] and a well defined N-dimensional covariance matrix $\boldsymbol{\Sigma}$ (in particular, non-singular),

$$\tilde{\mathbf{r}} \sim N(\tilde{\boldsymbol{\mu}}, \boldsymbol{\Sigma}) \tag{1}$$

Investor Prior Views

Our first step is to model the investor views. BL model considers views on expectations. In the normal market (1), this corresponds to statements on the parameter $\tilde{\boldsymbol{\mu}}$. Furthermore, BL focus on linear

[4] As they are not observable, one can only infer their probability distribution.
[5] We use boldface font for vectors to distinguish them from scalars.
[6] Unless otherwise specified, all returns refer to excess returns. The expected excess return of an asset is the expected return in the domestic currency minus the domestic cash return (also called risk free rate), given by $E[\tilde{\mathbf{r}}] - r_{free}$.
[7] The expected return is shorthand for $E[\tilde{\mathbf{r}}_{t+1} \mid I_t]$, where I_t refers to all information up to time t.

views[8] which can be relative as well as absolute. These may have direct implications for the securities he cover and also, owing to the dependence structure, indirect implications for those in the universe of securities not covered. In addition, considering (1), each view must assign a level of variance to quantify uncertainty around the estimate.

Let K be the total number of views, where $K \leq N$, $\mathbf{P}$ be a $K \times N$ matrix of view structure parameters whose rows are these assets (or portfolios) weights and $\mathbf{q}$ be a K-vector of the expected returns on these portfolios. The views can be expressed by

$$\mathbf{P}\tilde{\boldsymbol{\mu}} = \mathbf{q} + \tilde{\varepsilon} \tag{2}$$

where $\mathbf{P} \in \Re^{KxN}$ is known, $\mathbf{q} \in \Re^{K}$ is known and $\tilde{\boldsymbol{\mu}}$ is the (unknown but required) posterior vector of expected return estimates. $\tilde{\boldsymbol{\varepsilon}}$ is the unobservable vector of view estimation errors that is normally distributed as follows

$$\tilde{\boldsymbol{\varepsilon}} \sim N(\mathbf{0}, \boldsymbol{\Omega}) \tag{3}$$

where $\mathbf{0}$ is a vector of zeros and $\boldsymbol{\Omega} \in \Re^{KxK}$ is a diagonal variance matrix of view estimation errors, which, for simplicity, are considered independent across views. We can parameterize the prior distribution of expected returns as

$$\mathbf{P}\tilde{\mu} \sim N(\mathbf{q} \mid I_p, \boldsymbol{\Omega} \mid I_p) \tag{4}$$

where the location and dispersion parameters rely solely on the prior information I_p.

Market equilibrium returns

Secondly we define the equilibrium risk premiums as $\boldsymbol{\pi}$ [9]in terms of CAPM (or either in the sense of a value weighted index, a benchmark portfolio, for instance). Assuming that all investors share the same view and at that moment there is only one optimal portfolio, this portfolio is the one that contains all assets proportional to their capitalization weights, that is the market portfolio $\mathbf{w_m}$. The equilibrium risk premiums are such that the demand for these assets exactly equals to the outstanding supply ([Black (1989)]). Assuming the validity of CAPM, it follows that

$$\boldsymbol{\pi} = \boldsymbol{\beta}(E[r_m] - r_{free}) \tag{5}$$

[8] in terms of linear combinations of the securities expected returns.

[9] where $\boldsymbol{\pi} = E[\tilde{\mathbf{r}}] - r_{free}$

where

$$\boldsymbol{\beta} = \frac{Cov(\mathbf{r}, \mathbf{r}'\mathbf{w_m})}{\sigma_m^2} \tag{6}$$

where r_m is the global market return, r_{free} is the risk less cash return, $\boldsymbol{\beta}$ is a vector of N asset betas, $\mathbf{r}$ is also a vector of N asset estimated returns, $\mathbf{r}'\mathbf{w_m}$ is the market estimated return and σ_m^2 is the variance of the market returns. Let $\boldsymbol{\Sigma} = Cov(\mathbf{r}, \mathbf{r}')$, then

$$\boldsymbol{\pi} = \delta \boldsymbol{\Sigma} \mathbf{w}_m \tag{7}$$

where the average global risk aversion parameter is given by δ (and $\delta = \frac{E[r_m] - r_{free}}{\sigma^2}$).[10]

The expected returns $\tilde{\boldsymbol{\mu}}$ are considered to be random variables themselves with a probability distribution centered at the equilibrium returns and variance proportional to the covariance matrix of the returns. They are assumed to be normally distributed with the mean of $\boldsymbol{\pi}$

$$\tilde{\boldsymbol{\mu}} = \boldsymbol{\pi} + \tilde{\boldsymbol{\varepsilon}} \tag{8}$$

where $\tilde{\boldsymbol{\varepsilon}} \sim N(\mathbf{0}, \tau\boldsymbol{\Sigma})$. As the market is not necessarily in equilibrium, the assessment $\boldsymbol{\pi}$ suffers from errors. Moreover, the elements of $\tau\boldsymbol{\Sigma}$ should be smaller than those of $\boldsymbol{\Sigma}$ in a market demonstrating some level of semi strong form of market efficiency ([Fama (1965)]). Thus the parameter τ is an scalar between $(0,1)$.

The likelihood function of the data equilibrium returns given the investor prior beliefs, is thus given by

$$f(\boldsymbol{\pi} \mid \tilde{\boldsymbol{\mu}}, I_{eq}) \sim N(\tilde{\boldsymbol{\mu}}, \tau\boldsymbol{\Sigma} \mid I_{eq}) \tag{9}$$

where $\tilde{\boldsymbol{\mu}}$ is the unobservable mean and $\boldsymbol{\pi}$ and $\boldsymbol{\Sigma}$ are estimated encompassing all the equilibrium information I_{eq} contained in the distribution.

Using Bayes Theorem for the Estimation Model

At this point, we apply Bayes theory to blend the prior distribution and the likelihood function to create a new posterior distribution of the asset expected returns. It is more natural to think of $\boldsymbol{\pi}$ as the input of the quantitative investor, given it depends upon data. That was the reason why we defined its

[10] which is a positive scalar.

distribution as the likelihood function (or conditional distribution) and the conjugate prior distribution was represented by the investor particular views.

We can write the posterior expected return distribution, applying Bayes rule, as

$$f(\tilde{\boldsymbol{\mu}} \mid \boldsymbol{\pi}) = \frac{f(\boldsymbol{\pi} \mid \tilde{\boldsymbol{\mu}}) f(\tilde{\boldsymbol{\mu}})}{f(\boldsymbol{\pi})} \tag{10}$$

where $f(\boldsymbol{\pi} \mid \tilde{\boldsymbol{\mu}})$ is the conditional pdf of the data equilibrium return, upon the investor common beliefs, $f(\tilde{\boldsymbol{\mu}})$ is known as the prior pdf that expresses the investors' views and $f(\boldsymbol{\pi})$ represents the marginal pdf of equilibrium returns, a constant that will be absorbed into the integrating constant of the $f(\tilde{\boldsymbol{\mu}} \mid \boldsymbol{\pi})$

And by substituting the distributions (4) and (9) in (10) we get

$$\mathbf{f}(\tilde{\boldsymbol{\mu}} \mid \boldsymbol{\pi}) \sim N(\boldsymbol{\mu}_{BL}, \boldsymbol{\Sigma}_{BL}^{\mu}) \tag{11}$$

where the mean is given by

$$\boldsymbol{\mu}_{BL} = \left[(\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right]^{-1}\left[(\tau\boldsymbol{\Sigma})^{-1}\boldsymbol{\pi} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{q}\right] \tag{12}$$

and the covariance matrix is given by

$$\boldsymbol{\Sigma}_{BL}^{\mu} = \left[(\tau\boldsymbol{\Sigma})^{-1} + \mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P}\right] \tag{13}$$

This posterior covariance matrix is essentially the uncertainty in the posterior mean estimate about the actual mean and not the covariance of the returns itself. To compute the posterior covariance of returns, we need to add [11] the covariance of the estimate about the mean to the variance of the distribution about the estimate as

$$\boldsymbol{\Sigma}_{BL} \equiv \boldsymbol{\Sigma}_{BL}^{\mu} + \boldsymbol{\Sigma} \tag{14}$$

where $\boldsymbol{\Sigma}$ is the known covariance of returns and $\boldsymbol{\Sigma}_{BL}^{\mu}$ is the covariance of the posterior distribution about the true mean.

[11] Given that the error in the estimate of the mean return is independent of the covariance of the returns around the true mean.

Given the mean $\boldsymbol{\mu}_{BL}$ and the covariance matrix $\boldsymbol{\Sigma}_{BL}$, the optimal portfolio can be calculated by a standard mean-variance optimization method. Assuming a risk aversion parameter δ, the maximization problem can be written as:

$$\max_{\mathbf{w}\in\Re^n} \mathbf{w}'\boldsymbol{\mu}_{BL} - \frac{\delta}{2}\mathbf{w}'\boldsymbol{\Sigma}_{BL}\mathbf{w} \tag{15}$$

Calculating the first order condition we can get

$$\mathbf{w}^* = \frac{1}{\delta}\boldsymbol{\Sigma}_{BL}^{-1}\boldsymbol{\mu}_{BL} \tag{16}$$

where $\mathbf{w}^*$ is the vector of the optimal portfolio weights.

METHODOLOGY DETAILING

Our proposed model builds on the BL model for portfolio selection, basically adding two contributions. The first is that we introduce in the investor objective function a risk measure named expected tail loss, which is useful in portfolio selection context as it supports the benefits of diversification. The second is to allow investor views to be expressed in terms of linear inequalities among expected returns, which seems more natural in the practice of portfolio selection.

The mean-variance framework has some limitations, when the random outcome of assets follows a non-normal distribution. It has long been recognized that there are several conceptual difficulties using variance as a measure of risk. Quadratic utility functions displays undesirable properties of satiation as well as increasing absolute risk aversion ([Huang Litzenberger (1988)]). Furthermore, the assumption of elliptically symmetric return distributions is problematic. In practice we observe asymmetric return distributions which make variance an inadequate risk measure, as it equally penalizes desirable upside and undesirable downside returns ([Kroner Ng (1998)]). Motivated by those difficulties, alternative downside risk measures have been proposed and analyzed in the literature ([Bertsimas, Lauprete Samarov (2004)]). In recent years, financial practitioners have extensively used quantile-based risk measures, such as Value-at-Risk (VaR). However, for non-normal distributions, it is well known that VaR may have undesirable properties.[12] Also, VaR is difficult to optimize for discrete distributions, when it is calculated using scenarios. In this case, VaR is both a non convex and non smooth function of positions and also has multiple local extrema, which causes considerable difficulties in portfolio selection models ([Rockafellar (1970)]). Given those shortcomings of VaR, we decided to adopt another percentile risk measure, a coherent measure of risk in the sense of Artzner et al. [1999] called

[12] Artzner et al. [1999] propose axioms that risk measures (called coherent) should satisfy and show that VaR is not a coherent risk measure because it violates one of their axioms: the sub-additivity. In fact, it is coherent only when it is based on the standard deviation of normal distributions. (see the Appendix A)

expected tail loss (ETL).[13] [Rockafellar Uryasev (2000)] provide a simple algorithm for optimizing portfolios using a simulation-based expected tail loss measure. Their approach is the foundation for the analysis that follows.

Expected tail loss framework

Let $f(\mathbf{x},\mathbf{y})$ be the loss associated with the decision vector $\mathbf{x}$, to be selected from a certain subset $\mathbf{X} \in \Re^n$ and a random vector $\mathbf{y} \in \Re^m$. The vector $\mathbf{x}$ can be interpreted as a portfolio, where $\mathbf{X}$ is the set of available portfolios (subject to some constraints) and the vector $\mathbf{y}$ stands for the uncertainties that affect the portfolio results, such as market prices, for instance. For each $\mathbf{x}$, the loss $f(\mathbf{x},\mathbf{y})$ is a random variable having a distribution defined in $\Re$ induced by the underlying probability distribution of $\mathbf{y} \in \Re^m$ denoted by $\mathbf{p(y)}$.[14] The probability of $f(\mathbf{x},\mathbf{y})$ not exceeding a threshold ξ is given by

$$F(\mathbf{x},\xi) = \int_{f(\mathbf{x},\mathbf{y}) \le \xi} p(\mathbf{y})d\mathbf{y} \tag{17}$$

where $F(\mathbf{x},\xi)$ is the cumulative distribution function for the loss associated with $\mathbf{x}$. Thus $F(\mathbf{x},\xi)$ is nondecreasing with respect to ξ and it is assumed to be everywhere continuous with respect to ξ.

We will denote the VaR at the α percent probability level[15] by $\xi_\alpha(\mathbf{x})$ defined by

$$\xi_\alpha(\mathbf{x}) = min\{\xi \in \Re : F(\mathbf{x},\xi) \ge \alpha\}. \tag{18}$$

We then define the following function denoted by e_α [16] for the loss random variable associated with $\mathbf{x}$ and any specified probability level $\alpha \in (0,1)$:

$$e_\alpha(\mathbf{x}) = (1-\alpha)^{-1} \int_{f(\mathbf{x},\mathbf{y}) \ge \xi_\alpha(\mathbf{x})} f(\mathbf{x},\mathbf{y})p(\mathbf{y})d\mathbf{y} \tag{19}$$

where $e_\alpha(\mathbf{x})$ can be interpreted as the conditional expectation of the loss associated with $\mathbf{x}$ relative to that loss being $\xi_\alpha(\mathbf{x})$ or greater.

[13] The important cons of ETL are that it is more sensitive than VaR to estimation errors and its accuracy is heavily affected by the accuracy of the tail modeling.
[14] We will assume, for simplicity, that the distribution has density. [Rockafellar Uryasev (2000)] define ETL for general distributions.
[15] which is the lower bound that is reached with given probability.
[16] which is the expected loss assuming that the lower bound is reached.

ETL by simulation methods

Assets logarithmic returns are traditionally assumed to follow a normal distribution, implying excess kurtosis to be zero ([Alexander (1998), Jorion (2000)]). Under this assumption, $e_\alpha(\mathbf{r})$ can be calculated analytically. However, this assumption is too restrictive and seldom holds. Standard statistical tests suggest heavy tails for most of financial time series (Fama [1965] and Mandelbrot [1963] provide evidence of the excess kurtosis of asset returns in the 1960s). Therefore, in our study, we apply the following method to calculate the ETL in the objective function.

One important contribution of [Rockafellar Uryasev (2000)] was to suggest the use of an auxiliary objective function instead of $e_\alpha(\mathbf{x})$ that has better computational properties.17. They prove that both $\xi_\alpha(\mathbf{x})$ and $e_\alpha(\mathbf{x})$ can be described as particular cases of a general function Ψ_α defined by

$$\Psi_\alpha(\mathbf{x},\xi) = \xi + (1-\alpha)^{-1} \int_{\mathbf{y}\in\Re_m} [f(\mathbf{x},\mathbf{y})-\xi]^+ p(\mathbf{y})d\mathbf{y} \tag{20}$$

where $[\tau]^+ = max\{\tau,0\}$.

One of its very desirable features is that, as a function of ξ, $\Psi_\alpha(\mathbf{x},\xi)$ is convex and continuously differentiable. And the $e_\alpha(\mathbf{x})$ of the loss associated with any $\mathbf{x}\in\mathbf{X}$ can be determined from the formula

$$e_\alpha(\mathbf{x}) = \min_{\xi\in\Re} \Psi_\alpha(\mathbf{x},\xi) \tag{21}$$

where the set of the values of ξ for which the minimum is attained is a nonempty, closed, bounded interval and the $\xi_\alpha(\mathbf{x})$ is given by

$$\xi_\alpha(\mathbf{x}) = \text{left endpoint of argmin}_{\xi\in\Re}\, \Psi_\alpha(\mathbf{x},\xi) \tag{22}$$

Therefore, minimizing $e_\alpha(\mathbf{x})$ of the loss associated with $\mathbf{x}$ over all $\mathbf{x}\in\mathbf{X}$ is equivalent to minimize $\Psi_\alpha(\mathbf{x},\xi)$ over all $(\mathbf{x},\xi)\in\mathbf{X}x\Re$

$$\min_{x\in X} e_\alpha(\mathbf{x}) = \min_{(\mathbf{x},\xi)\in\mathbf{X}x\Re} \Psi_\alpha(\mathbf{x},\xi) \tag{23}$$

[17] The expression (19) involves multidimensional integration. Such evaluation is computationally prohibitive above the fourth dimension.

As it is not necessary, for the purpose of determining the vector $\mathbf{x}$, to work directly with the function $e_\alpha(\mathbf{x})$ in (19), we will use this technique to minimize the simulation-based expected shortfall.

The integral in equation (20) can be approximated by sampling the probability distribution of $\mathbf{y}$ according to its density $p(\mathbf{y})$. Supposing it generates a collection of vectors $\mathbf{y_1}, \mathbf{y_2}, \ldots \mathbf{y_J}$, its approximation is calculated as follows:

$$\int_{\mathbf{y}\in\Re_m} [f(\mathbf{x},\mathbf{y})-\xi]^+ p(\mathbf{y})d\mathbf{y} \approx \sum_{j=1}^{J} pr_j [f(\mathbf{x},\mathbf{y})-\xi]^+ \tag{24}$$

where pr_j are probabilities of scenarios $\mathbf{y_j}$.[18]

Further after discretization, by using dummy variables $z_j, j=1,2,\ldots J$,the approximated function can be replaced by the linear function:

$$\xi + (1-\alpha)^{-1}\sum_{j=1}^{J} pr_j z_j, \tag{25}$$

and the set of linear constraints:

$$z_j \geq f(\mathbf{x},\mathbf{y})-\xi, z_j \geq 0, j=1,\ldots,J, \xi\in\Re \tag{26}$$

To get to the optimization problem we need to define the loss function and the risk and value constraints. Let us consider a portfolio composed of n different instruments among which there is one risk free asset (cash or bank account). Let $\mathbf{x^0}=(x_1^0, x_2^0, \ldots x_n^0)'$ be the positions in the initial portfolio and $\mathbf{y^0}=(y_1^0, y_2^0, \ldots y_n^0)'$ the initial prices for the instruments. The inner product $\mathbf{y}^{0'}\mathbf{x}^0$ corresponds to the initial portfolio value. Let $\mathbf{x}=(x_1, x_2, \ldots x_n)'$ be the positions in the optimized portfolio (that we intend to calculate using our algorithm) and $\mathbf{y}=(y_1, y_2, \ldots y_n)'$ the scenario dependent prices for each instrument. The loss function can be calculated as

$$f(\mathbf{x},\mathbf{y};\mathbf{x^0},\mathbf{y^0}) = \mathbf{y^0}'\mathbf{x^0} - \mathbf{y}'\mathbf{x} \tag{27}$$

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[18] If the loss function is linear with respect to $\mathbf{X}$, then the function $\Psi_\alpha(\mathbf{x},\xi)=\xi+(1-\alpha)^{-1}\sum_{j=1}^{J} pr_j[f(\mathbf{x},\mathbf{y})-\xi]^+$ is convex and piecewise linear, which is the matter of our case study.

which is linear and convex in $\mathbf{x}$.

We will consider a practical issue to define the risk constraint. We calculate the portfolio ETL at different confidence levels, considering ϖ as the percentage of the initial portfolio value that the investor is allowing for risk exposure. And as the loss function is convex in $\mathbf{x}$, therefore, $e_{\alpha}(\mathbf{x})$ function is also convex in $\mathbf{x}$. The set of linear constraints is defined as

$$\xi + (1-\alpha)^{-1}\sum_{j=1}^{J} pr_j z_j \le \varpi \sum_{i=1}^{n} y_i^0 x_i^0 , \qquad (28)$$

$$\begin{aligned} & z_j \ge \sum_{i=1}^{n} (y_i^0 x_i^0 - y_{ij} x_i) - \xi , \\ & z_j \ge 0,\ j = 1,\dots,J,\ \xi \in \Re \end{aligned} \qquad (29)$$

And finally, in our optimization problem the individual must be fully invested, although short positions are allowed. We can define this as the budget constraint: $\mathbf{x}'\iota = 1$, where ι corresponds to the n-vector of ones.

In the above constrained minimization problem we can observe that both the objective function and the constrains are linear functions of the decision variable. Since all linear functions are convex, it may be solved using a LP-solver.

CASE STUDY: PORTFOLIO OF BOVESPA STOCKS

We now proceed with a case study to calculate optimal portfolios of Brazilian stocks combining a more appropriate risk measure with scenario generation procedures, considering investors subjective opinions about expected returns. We have two main purposes in the research. The first is to test the value added by BL approach to the traditional mean-variance optimal portfolio composed of Brazilian equities. In this step we consider the analytic framework proposed by Black and Litterman. We calculate the mean vector for the expected returns as well as the posterior covariance matrix of returns. We consider them as the inputs in the MV model and generate the optimal portfolio for the analytic BL model. The second purpose is to substitute the variance by the expected tail loss in the objective function. We first optimize for the mean-ETL model using stocks returns simulated from a multivariate Gaussian distribution[19]. In the next step, to consider the investors' views in the optimal portfolio, we substitute the historical data by the BL calculated expected returns and posterior covariance matrix as

[19] We have used historical data (mean and covariance matrix) for scenarios generation. Although it is well documented in many academics studies that historical returns provide very little information on the actual returns ([Jorion (2000), Michaud (1989)]), it is still the most applied method by practitioners.

inputs in the simulation-based analysis. Finally, we backtest the optimized portfolios[20] to evaluate their performances out of sample. The models suggested bellow must be considered as one period of the multi-period investment models to be used in a realistic environment. The optimization algorithm is implemented and solved using Matlab.

How to create the investor views

We choose to describe the investor views using a realistic approach. Usually, equity research analysts use a very well studied measure as a criteria to impose different views over the analyzed stocks. As documented since 1960 by Nicholson and later by Basu, Sanjoy [1977, 1983] and Jaffe, Jeffrey, Keim, Westerfield [1989] it was shown that stocks with high earnings/price ratios (or low P/E ratios) earned significantly higher returns than stocks with low earnings/price ratios. And further, they showed that this behavior is not just observed among small cap stocks. In this way, we adopted the earnings/price ratio as measure to input the investor views to better predict future stock returns. Our database covers the time period ranging from 29 December 2005 to 30 June 2011. We collect daily prices for Brazilian stocks, adjusted for all splits and dividends in local currency and inflation adjusted, from Economatica database system. As standard, we ignored weekends and holidays and concentrate our analysis on trading days. The sample stocks must exist during all the studied period. After the above filtering, there remains 45 stocks to be considered in this empirical research.

To construct the set of investor views we define three different views. View 1 is set as an absolute view. The investor decides to buy an equal weighted portfolio composed of the ten stocks traded at Bovespa with the smallest P/E ratio at the last trading day of each year (2005 to 2010)[21]. This portfolio is expected to outperform the Bovespa index for the subsequent year. We define its total expected return as the Brazilian government interest rate plus the literature equity risk premium (considered to be 5 percent per year) plus the portfolio expected alpha return, defined as 10 percent per year.

On the other hand, view 2 is set as a relative view. The investor decides to hold a long and short equal weighted portfolio. He buys five stocks with the lowest P/E ratios and sells five stocks with the highest P/E ratios, at the end of each year, considering the sample studied. And once more, at the end of each year, this portfolio composition may vary, depending on the updated P/E ratios. This long and short market neutral portfolio of views is expected to generate an absolute return equal to the Brazilian government interest rate of return[22]. View 3 is also set as a relative view, but, at this stage, the investor doesn't want to be exposed to industry or macroeconomic risks. He holds a long and short equal

[20] The basic model considered in this study is the Markowitz model (mean-variance) based on historical mean returns and covariance matrix.

[21] This portfolio is rebalanced at the end of each year.

[22] This is the most widespread benchmark portfolio managers set as target to long and short equity strategies in Brazil. We disregard stocks rental premium or costs in our analysis.

weighted portfolio considering stocks of the same industry. He composes three different portfolios, one in each industry studied (energy, finance and mining and steel). He buys the two stocks with the lowest P/E ratios and sells the two stocks with the highest P/E ratios, at the end of each year, in each of those industries. Also, at the end of each year, this portfolio composition may vary, depending on the updated P/E ratios. This long and short industry neutral portfolio of views is expected to generate an absolute return equal to half the Brazilian government interest rate of return[23].

The Equilibrium portfolio hypothesis

Using the approach of liquidity to arrive at a fair price, we decide to consider the Bovespa Index composition as the appropriate proxy for the equilibrium portfolio for each period. The Bovespa index is the most widely quoted share index for Brazil and the most important indicator of the stock market in Brazil, representing the average behavior of prices of the main stocks at the Sao Paulo Stock Exchange. To simplify, we decide to re-balance the portfolio considering 80 percent of its composition at each year. Also, as our sample consists of 45 stocks that existed for the whole period, we rebalanced again considering the relative weights in those 45 stocks. This final portfolio will be hereon our market equilibrium portfolio.

We combine the three views above with our equilibrium portfolio to calculate the BL expected mean and posterior covariance matrix to be considered as the inputs in the optimization problem.

We break the sample in 6 subsamples for estimation (2005, 2006, 2007, 2008, 2009, 2010) and, for each subsample, the optimal portfolios are rebalanced at the end of each calendar year and its performance tested for the subsequent year (out of sample).

RESULT ANALYSIS

We propose different returns and risk measures to analyze the optimal portfolios generated by the methodologies presented above. Hereon, we compare the out of sample results for the following models: analytic mean-variance using historical returns (MV), mean variance optimization using Black and Litterman expected returns and posterior covariance matrix as inputs (BLMV), multivariate Gaussian returns simulated using historical data as inputs in an optimization problem defined as minimizing the expected tail loss function (METL) and finally multivariate Gaussian returns simulated using Black and Litterman expected returns and posterior covariance matrix as inputs in an optimization problem defined as minimizing the expected tail loss function (BLETL).

From figure 1 we can analyze the portfolio composition in terms of concentration and maximum assumed positions (long or short). For MV and BLMV models, we find optimal portfolios more concentrated in few stocks (both for long and short positions), except for year estimation 2009. When we analyze the year of 2008, we can observe that the MV optimal portfolio allocate 40% of its

[23] This is also a practice in the Brazilian market. As the investor, trading this strategy, is only exposed to company specific risks, its expected return is defined to be less than the long and short market neutral portfolio.

composition in one stock, which seems very concentrated. In fact, all the results obtained for this year must be analyzed considering the stock market crash generated by the sub-prime mortgage crisis in the US. Analyzing the portfolio composition histogram for the year of 2008, we notice that the BLETL model presented the less concentrated optimal portfolio having 67% of its total portfolio in positions of size less than 10% each.

Figure 1: Model portfolio composition.

		MV	BL_MV	M_ETL	BL_ETL
Estimation period: year calendar 2005					
Max weight		37.5%	38.6%	33.6%	33.4%
Min weight		-21.1%	-19.8%	-13.8%	-16.9%
	< -20%	2%	0%	0%	0%
	-20% ≤ w < -10%	4%	11%	4%	7%
	-10% ≤ w < -5%	2%	13%	9%	9%
Weights Histogram	**-5% ≤ w < 5%**	69%	33%	53%	53%
	5% ≤ w < 10%	13%	24%	22%	18%
	10% ≤ w < 20%	7%	16%	9%	11%
	≥20%	2%	2%	2%	2%
Estimation period: year calendar 2006					
Max weight		24.4%	28.5%	24.8%	18.7%
Min weight		-14.7%	-11.8%	-13.3%	-16.8%
	< -20%	0%	0%	0%	0%
	-20% ≤ w < -10%	2%	9%	2%	7%
	-10% ≤ w < -5%	13%	11%	16%	9%
Weights Histogram	**-5% ≤ w < 5%**	53%	49%	53%	49%
	5% ≤ w < 10%	18%	16%	20%	20%
	10% ≤ w < 20%	11%	9%	7%	16%
	≥20%	2%	7%	2%	0%
Estimation period: year calendar 2007					
Max weight		25.7%	37.1%	36.9%	32.0%
Min weight		-32.6%	-26.9%	-32.7%	-22.1%
	< -20%	4%	2%	2%	2%
	-20% ≤ w < -10%	4%	18%	2%	7%
	-10% ≤ w < -5%	4%	11%	18%	11%
Weights Histogram	**-5% ≤ w < 5%**	53%	24%	42%	44%
	5% ≤ w < 10%	16%	24%	16%	13%
	10% ≤ w < 20%	13%	16%	18%	18%
	≥20%	4%	4%	2%	4%
Estimation period: year calendar 2008					
Max weight		40.0%	37.7%	35.2%	36.5%
Min weight		-24.2%	-23.1%	-23.9%	-25.9%
	< -20%	2%	9%	4%	2%
	-20% ≤ w < -10%	11%	16%	13%	9%
	-10% ≤ w < -5%	9%	7%	9%	16%
Weights Histogram	**-5% ≤ w < 5%**	42%	27%	31%	38%
	5% ≤ w < 10%	11%	11%	13%	13%
	10% ≤ w < 20%	18%	20%	22%	18%
	≥20%	7%	11%	7%	4%
Estimation period: year calendar 2009					
Max weight		29.4%	43.3%	29.6%	54.8%
Min weight		-21.8%	-40.7%	-16.7%	-43.2%
	< -20%	2%	2%	0%	2%
	-20% ≤ w < -10%	9%	18%	11%	9%
	-10% ≤ w < -5%	9%	7%	4%	16%
Weights Histogram	**-5% ≤ w < 5%**	51%	33%	53%	36%
	5% ≤ w < 10%	13%	13%	16%	18%
	10% ≤ w < 20%	7%	13%	11%	13%
	≥20%	9%	13%	4%	7%
Estimation period: year calendar 2010					
Max weight		21.6%	33.8%	17.4%	21.4%
Min weight		-18.8%	-38.8%	-22.4%	-13.9%
	< -20%	0%	9%	2%	0%
	-20% ≤ w < -10%	7%	9%	4%	7%
	-10% ≤ w < -5%	7%	9%	11%	7%
Weights Histogram	**-5% ≤ w < 5%**	47%	33%	49%	58%
	5% ≤ w < 10%	27%	13%	13%	20%
	10% ≤ w < 20%	11%	13%	20%	7%
	≥20%	2%	13%	0%	2%

We can analyze the optimal portfolios return and risk statistics in figure 2. From the risk perspective, both in terms of volatility, maximum draw down and value-at-risk measures, the 4 models presented a lower level of risk when compared to the Bovespa index. Highlighting the crisis year of 2008, we can verify that while the Bovespa index presented a maximum draw down of almost 60% in absolute terms, the BLETL model presented a much smaller loss of 39,7% for the same period.

In terms of accumulated performance, the BLETL model also presented the lowest loss. Its portfolio decreased 29,7% in value when compared to the Bovespa index that decreased 41,2%. Except for years 2007 and 2009 (out of sample), the optimal portfolios generated by the 4 different models presented better risk adjusted performances when compared to the Bovespa index. And during years 2007 and 2009, it is important to notice that the Bovespa index out-performances were followed by higher risk measures (both volatility, value-at-risk as well as maximum draw downs). When we compare the models MV and BLMV we can also verify, except for the year 2008, an improvement in absolute return as measured by accumulated, average or percentile return. This behavior in performance is not verified when we compare models METL and BLETL. On the other hand, from the risk perspective, optimal portfolios generated by BLETL model presented systematically lower losses both in terms of value-at-risk and maximum draw downs, except for year 2010. Finally, from the Sharpe index measure we can conclude that, except for year 2009, the BLMV model presented the best risk adjusted return.

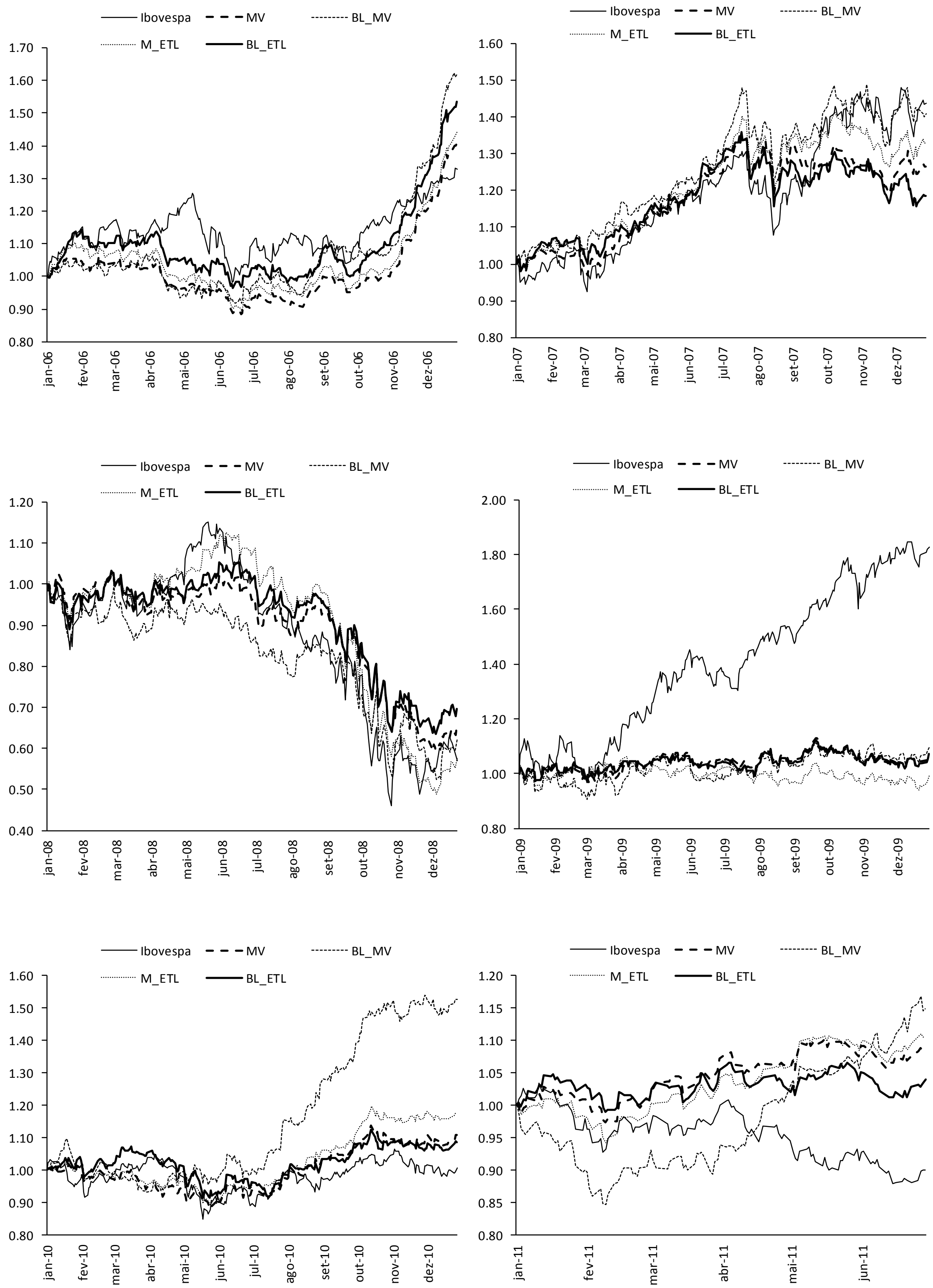
Figure 2: Performance and risk analysis (out of sample)

	Ibovespa	MV	BL_MV	M_ETL	BL_ETL
Test period: year calendar 2006					
Acc. Return	32.9%	40.7%	62.1%	44.3%	53.5%
Average Return	29.2%	35.0%	49.5%	37.6%	43.9%
Max	4.8%	3.3%	4.3%	3.2%	4.2%
Min	-4.6%	-3.9%	-4.0%	-4.3%	-5.1%
Perc 95%	2.6%	1.8%	2.3%	2.0%	2.0%
Perc 5%	-2.5%	-1.3%	-1.8%	-1.6%	-1.5%
Volat %pd	24.2%	16.1%	20.1%	17.1%	18.5%
Max DD	-21.8%	-17.0%	-14.1%	-18.7%	-15.7%
Sharpe Index	0.62	1.29	1.75	1.37	1.61
Test period: year calendar 2007					
Acc. Return	43.6%	26.6%	41.0%	33.0%	18.4%
Average Return	37.3%	24.2%	35.3%	29.3%	17.4%
Max	4.8%	2.8%	4.0%	3.5%	3.5%
Min	-6.9%	-5.0%	-4.3%	-4.3%	-4.2%
Perc 95%	2.6%	1.7%	2.1%	1.9%	1.9%
Perc 5%	-3.2%	-2.0%	-2.0%	-2.3%	-2.2%
Volat %pd	27.5%	17.7%	21.5%	19.4%	19.7%
Max DD	-17.4%	-11.9%	-17.0%	-15.2%	-14.9%
Sharpe Index	0.95	0.73	1.12	0.93	0.31
Test period: year calendar 2008					
Acc. Return	-41.2%	-35.5%	-38.9%	-42.4%	-29.7%
Average Return	-53.8%	-44.4%	-49.8%	-55.9%	-35.7%
Max	13.7%	10.7%	13.2%	12.7%	10.1%
Min	-12.1%	-6.9%	-12.2%	-10.1%	-7.3%
Perc 95%	4.6%	2.8%	3.7%	3.5%	3.3%
Perc 5%	-5.4%	-3.6%	-3.9%	-4.7%	-3.5%
Volat %pd	52.3%	34.2%	41.8%	40.2%	36.1%
Max DD	-60.0%	-42.1%	-47.7%	-56.6%	-39.7%
Sharpe Index	neg	neg	neg	neg	neg
Test period: year calendar 2009					
Acc. Return	82.7%	7.7%	9.7%	-0.7%	7.4%
Average Return	61.7%	7.6%	9.5%	-0.7%	7.3%
Max	6.9%	2.8%	4.4%	3.1%	2.8%
Min	-5.4%	-3.8%	-5.5%	-4.7%	-3.5%
Perc 95%	3.3%	1.9%	2.5%	2.0%	1.9%
Perc 5%	-3.0%	-2.0%	-2.7%	-2.3%	-2.1%
Volat %pd	31.4%	19.2%	25.1%	20.7%	18.7%
Max DD	-15.3%	-9.7%	-11.7%	-11.6%	-9.2%
Sharpe Index	1.66	neg	0.00	neg	neg
Test period: year calendar 2010					
Acc. Return	1.0%	12.3%	55.4%	19.3%	10.7%
Average Return	1.1%	11.8%	45.0%	18.0%	10.4%
Max	4.0%	2.8%	3.8%	2.5%	4.0%
Min	-4.8%	-2.1%	-3.7%	-2.4%	-3.3%
Perc 95%	2.0%	1.5%	1.9%	1.5%	1.9%
Perc 5%	-2.1%	-1.4%	-1.7%	-1.3%	-1.7%
Volat %pd	20.4%	14.4%	18.9%	13.8%	17.5%
Max DD	-18.9%	-12.7%	-15.1%	-12.7%	-14.8%
Sharpe Index	neg	0.07	1.80	0.53	neg
Test period: year calendar 2011					
Acc. Return	-10.0%	8.9%	14.8%	10.5%	4.0%
Average Return	-21.5%	17.5%	28.3%	20.4%	8.0%
Max	1.9%	2.8%	3.3%	2.3%	2.4%
Min	-2.4%	-2.3%	-2.9%	-2.4%	-2.1%
Perc 95%	1.6%	1.4%	1.9%	1.4%	1.4%
Perc 5%	-1.9%	-1.1%	-2.1%	-1.3%	-1.1%
Volat %pd	16.2%	12.5%	19.0%	12.8%	12.6%
Max DD	-199.7%	-5.3%	-15.4%	-6.9%	-5.6%
Sharpe Index	0.46	0.53	0.91	0.74	neg

From figure 3 we can analyze the evolution of accumulated return for the different portfolios. For the whole sample period, we can conclude that BLMV model outperformed the MV traditional model. This implies that the methodology we used to create our views about expected returns was satisfactory. We could add value with those views and get optimal portfolios that performed better out of sample for the analyzed period. This result is also positive when we compare the BLMV model with the Bovespa index. Except for year 2009 when the index recovered more than 80% and the models didn't follow this movement[24], the BLMV model outperformed the index with major magnitude.

[24] One possible explanation for this behavior is that our models only consider allocating among the 45 selected stocks whereas the Bovespa index composition depends only on a liquidity rule.

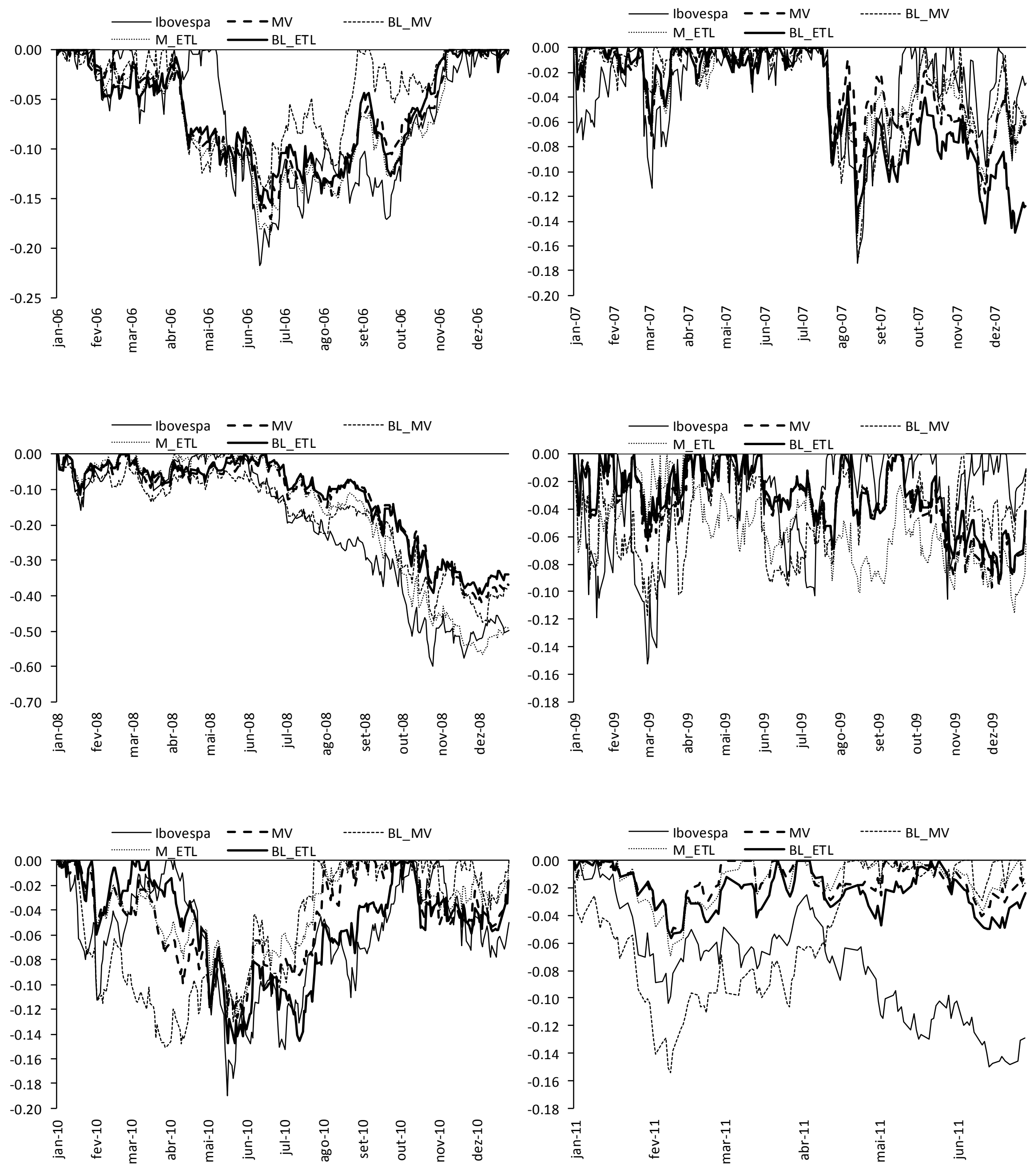
Figure 3: Accumulated performance analysis (out of sample)



Fernandes, Betina; Fernandes, Cristiano; Street, Alexandre (2012). An asset allocation model with inequalities constraints and coherent risk measure: an application to Brazilian equities. Revista de Finanças Aplicadas. Publicado em 21/01/2013, pp.1-27.

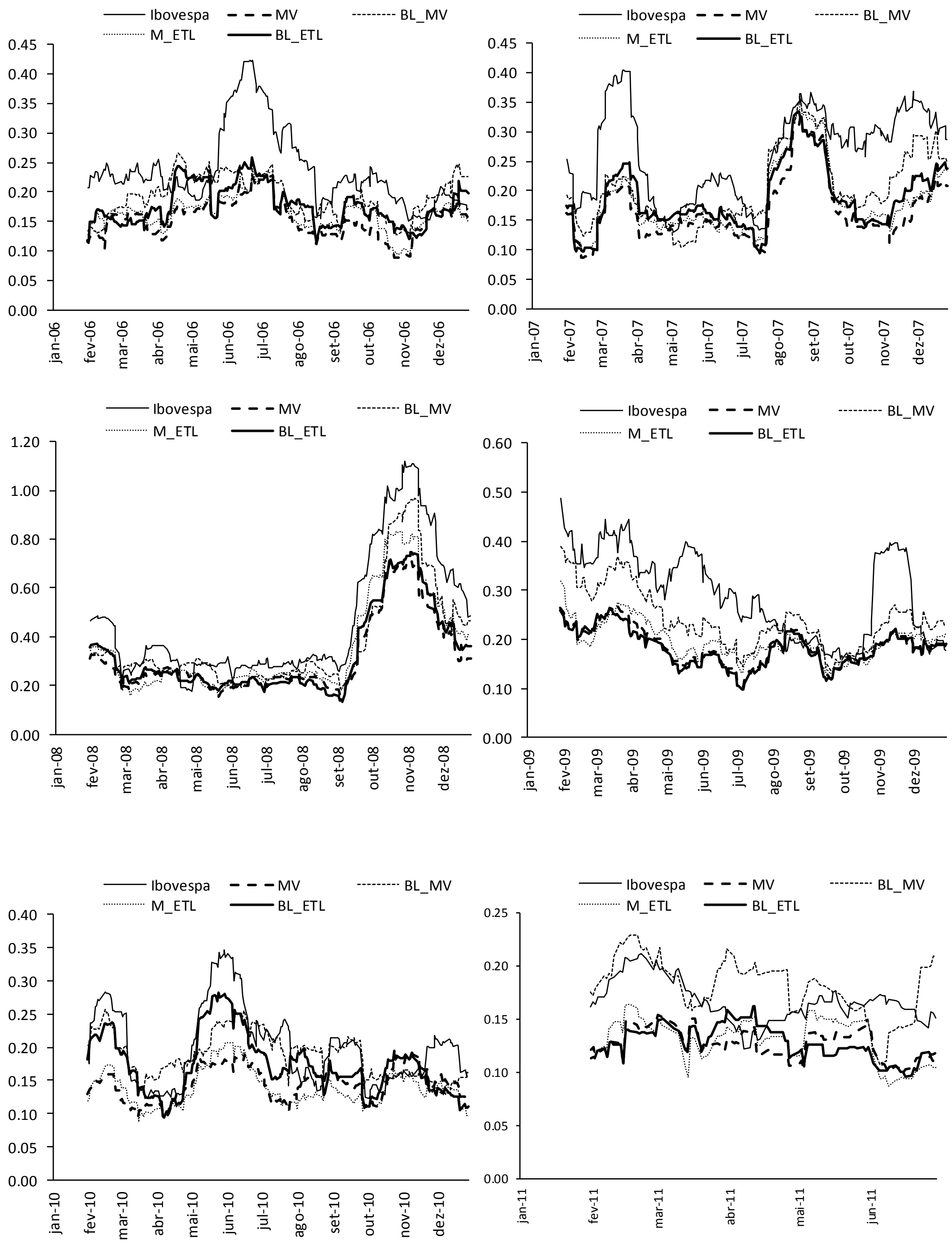
In figure 4 we can compare the evolution of draw down risk measure among the models and the Bovespa index. From the graphs we can conclude that the Bovespa index tends to present higher draw downs when compared to the proposed models. This behavior is even greater during crisis periods as October 2008, may 2010 and June 2011.

Figure 4: draw down analysis (out of sample)



To continue our analysis from risk perspective, in figure 5 we have the evolution of volatility risk measure among the models and the Bovespa index. Again we can conclude that the Bovespa index tends to present higher and more persistent volatility when compared to the proposed models.

Figure 5: Volatility analysis (moving 21 trading days)



FINAL CONSIDERATIONS

Practitioners are well aware that asset returns are not normally distributed and that investor preferences often go beyond mean and variance; however, the implications for portfolio choice are not well known. In this study we could obtain insights into optimal asset allocations whether defining different objective functions (varying its risk measure) and the inputs of the problem (expected returns and covariance matrix). In a series of controlled optimizations, we compare optimal asset allocation weights obtained from the traditional mean-variance models (MV) with those from mean-expected tail loss (METL), analytic Black-Litterman approach (BLMV) and our proposed model (BLETL). We find that for the Brazilian equity market the BLMV model outperformed the MV traditional model, which shows that the methodology implemented to create our views about expected returns was successful. And also that optimal portfolios generated by BLETL model presented systematically lower losses both in terms of value-at-risk and maximum draw downs when compared to the other specifications and to the Bovespa index.

While we believe that we have made progress on important issues in portfolio selection and that our exercise is applied to out-of sample portfolio choice problem, there are at least two limitations to our approach. First, our information is restricted to past market data (returns and earnings per share) which means that investors make decisions based on past information and do not use other conditioning information that could infer about the state of the economy. Second, our exercise is a static portfolio selection problem. There is an extensive literature that considers the asset allocation problem as dynamic, which allows for portfolio weights to change with investment horizon, labor income and other economic variables.

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APPENDIX: A NOTE ON COHERENT RISK MEASURES

Artzner et al.[1999] propose four axioms which every measure of risk should satisfy. Let $\mathbf{X}$ be a stochastic variable on the set $\boldsymbol{\psi}$, where $\boldsymbol{\psi}$ is the set of all possible outcomes (all possible risks). A risk measure, given by $\boldsymbol{\rho}$ is the mapping

$$\boldsymbol{\rho} : \boldsymbol{\psi} \rightarrow \Re \quad (30)$$

which means that if $\mathbf{X}$ is a possible outcome then $\boldsymbol{\rho}(\mathbf{X})$ is the risk of the random variable $\mathbf{X}$. Suppose a scalar $\lambda \in \Re$, they call $\rho(\mathbf{X})$ a coherent risk measure if the following properties are fulfilled:

1. Monotonicity: $\forall \mathbf{X}_i, \mathbf{X}_j \in \psi$ such that $\mathbf{X}_i \leq \mathbf{X}_j$ we have $\rho(\mathbf{X}_j) \leq \rho(\mathbf{X}_i)$,
2. Positive homogeneity: if $\lambda \geq 0$ and $\mathbf{X} \in \psi$ then $\rho(\lambda \mathbf{X}) = \lambda \rho(\mathbf{X})$,
3. Sub-additivity: if $\mathbf{X}_i, \mathbf{X}_j \in \psi$ then $\rho(\mathbf{X}_i + \mathbf{X}_j) \leq \rho(\mathbf{X}_i) + \rho(\mathbf{X}_j)$,
4. Translation Invariance: if $\mathbf{X} \in \psi$ and $\lambda \in \Re$ then $\rho(\mathbf{X} + \lambda) = \rho(\mathbf{X}) - \lambda$.

The first property gives that if an asset $\mathbf{X}_j$ is worth more than other asset $\mathbf{X}_i$, then the risk of $\mathbf{X}_i$ is always greater. The second property says that the amount of risk is also dependent on the size of the position. In practice, the most desirable property is the sub-additivity. It gives an investor incentive to diversify her portfolio, since it ensures that the risk of two assets is less or equal to the risk of the the two separate assets. And any risk measure that possesses property 2 and 3 is said to be convex. Choosing $\lambda \in (0,1)$ we can write

$$\rho(\lambda \mathbf{X}_i + (1-\lambda)\mathbf{X}_j) \leq \rho(\lambda \mathbf{X}_i) + \rho((1-\lambda)\mathbf{X}_j) = \lambda \rho(\mathbf{X}_i) + (1-\lambda)\rho(\mathbf{X}_j) \quad (31)$$

The above definition rules out as incoherent, under general distributional assumptions, risk measures based on variance (violates property 4), on VaR (violates property 3) and on semi-variance (violates property 4). However, when the asset returns have elliptically symmetric distributions, all of these are coherent.